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CONTINUOUS FAMILIES OF ISOSPECTRAL RIEMANNIAN METRICS WHICH ARE NOT LOCALLY ISOMETRIC

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Introduction

Two compact Riemannian manifolds are said to be *isospectral* if the associated Laplace-Beltrami operators, acting on smooth functions, have the same eigenvalue spectrum. If the manifolds have boundary, we specify *Dirichlet* or *Neumann* isospectrality depending on the boundary conditions imposed on the eigenfunctions.

Numerous examples of isospectral compact manifolds have been constructed; see, for example, [4], [5], [7], [12], [13], [14], [15], [16], [17], [19], [24] and [26] or the survey articles [1], [2], [6], and [9]. Until recently however, all known examples of isospectral manifolds were locally isometric, though not globally isometric. In particular, the closed isospectral manifolds had a common cover. Then Z. Szabó [25] gave a construction of pairs of isospectral compact manifolds with boundary which are not locally isometric, and the first author [10], [11] constructed pairs of isospectral closed Riemannian manifolds which are not locally isometric. Szabo pointed out that the curvature operators of these isospectral manifolds have different eigenvalues, thus identifying a specific local invariant which is not spectrally determined.

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The first goal of this paper is to exhibit *continuous* families of isospectral Riemannian manifolds which are not locally isometric, i.e., we continuously deform the Riemannian metrics in such a way that the local geometry changes but the Laplace spectrum remains invariant. In fact we prove:

Theorem 0.1. *Let B be a ball in $\mathbf{R}^m$, $m \geq 5$, and let T^r be a torus of dimension $r \geq 2$. Then there exist continuous d-parameter families of Riemannian metrics on the compact manifold $B \times T^r$ which are both Dirichlet and Neumann isospectral but not locally isometric. Here d is of order at least $O(m^2)$.*

A precise lower bound on d is given in Theorem 2.2.

We also consider closed manifolds. Here we have not been able to construct examples of continuous isospectral deformations in which the metrics are not locally isometric. However, we do construct new examples of pairs of isospectral closed manifolds which are not locally isometric.

Next we examine the local geometry of the isospectral manifolds. Since all the manifolds considered in this paper are locally homogeneous, the curvature does not vary from point to point. In particular the eigenvalues of the Ricci tensor are constant functions on each manifold. We exhibit specific examples of isospectral deformations of manifolds with boundary for which the eigenvalues of the Ricci tensor deform non-trivially. Similarly, we exhibit pairs of isospectral closed manifolds whose Ricci tensors have different eigenvalues. These examples illustrate for the first time that the Ricci curvature is not spectrally determined.

The paper is organized as follows:

In §1, we give a method for constructing isospectral metrics on $B \times T^r$ which are not locally isometric. The construction reduces to a problem in linear algebra:

(P) Find pairs of r-dimensional subspaces of $so(m)$ and an isomorphism between these subspaces such that corresponding elements have the same spectrum but the two subspaces are not conjugate by any orthogonal transformation.

As will be explained in §1, each subspace of $so(m)$ gives rise to a two-step nilpotent Lie algebra with an inner product and thus to a simply-connected nilpotent Lie group with a left-invariant Riemannian metric. The non-conjugacy condition in (P) guarantees that the resulting pair of nilpotent Lie groups with metrics are not locally isometric. The manifolds in Theorem 0.1 are domains with boundary in these nilpotent Lie groups (more precisely, in nilpotent Lie groups covered by these simply-connected ones). We show that the spectral condition in (P) guarantees the isospectrality of these compact manifolds with boundary. We end §1 with a 7-dimensional example.

In §2, we give an explicit construction of continuous families of 2-dimensional subspaces of so(6) satisfying pairwise the condition (P) described above. Moreover, we show that for $m = 5$ and for $m \geq 7$, generic two-dimensional subspaces of $so(m)$ belong to d-parameter families of subspaces which satisfy pairwise the condition (P), where d is of order $O(m^2)$. This completes the proof of Theorem 0.1.

In §3 we consider *nilmanifolds*, i.e., closed manifolds arising as quotients $\Gamma\backslash G$ of nilpotent Lie groups by discrete subgroups, endowed with Riemannian metrics induced from left-invariant metrics on G. We generalize the construction given in [10], [11] of isospectral nilmanifolds. We construct seven and eight-dimensional examples of isospectral nilmanifolds by taking quotients of suitable pairs of the simply-connected nilpotent Lie groups occurring in the examples in §1 and §2.

§4 examines the curvature of the various examples, in particular showing that many of the isospectral manifolds have different Ricci curvature.

An appendix supplies a proof of a result needed in §3.

We wish to acknowledge Zoltan Szabó's beautiful work [25] which inspired Theorem 0.1.

1. Lie algebra criteria for local isometry and isospectrality

A left-invariant Riemannian metric g on a connected Lie group G corresponds to an inner product $< \cdot\, , \cdot >$ on the Lie algebra $\mathfrak{g}$ of G. We will call the pair $(\mathfrak{g}, < \cdot\, , \cdot >)$ a *metric Lie algebra*. Recall that G is said to be *two-step nilpotent* if $[\mathfrak{g}, \mathfrak{g}]$ is a non-zero subspace of the center of $\mathfrak{g}$. Letting $\mathfrak{z} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{v} = \mathfrak{z}^{\perp}$ relative to $< \cdot\, , \cdot >$, we can then define an injective linear map $j : \mathfrak{z} \to \mathrm{so}(\mathfrak{v}, < \cdot\, , \cdot >)$ by

$$\langle j(z)x, y\rangle = \langle [x, y], z\rangle \text{ for } x,\ y \in \mathfrak{v}, z \in \mathfrak{z}. \tag{1.1}$$

Conversely, given any two finite dimensional real inner product spaces $\mathfrak{v}$ and $\mathfrak{z}$ along with a linear map $j : \mathfrak{z} \to \mathrm{so}(\mathfrak{v})$, we can define a metric Lie alegbra $\mathfrak{g}$ as the orthogonal direct sum of $\mathfrak{v}$ and $\mathfrak{z}$ with the alternating bilinear bracket map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ defined by insisting that $\mathfrak{z}$ be central in $\mathfrak{g}$ and using (1.1) to define $[x, y]$ for $x, y \in \mathfrak{v}$. Then $\mathfrak{g}$ is two-step nilpotent if j is non-zero, and $\mathfrak{z} = [\mathfrak{g}, \mathfrak{g}]$ if j is injective. We will always assume j is injective.

In the sequel, we will fix finite dimensional inner product spaces $\mathfrak{v}$ and $\mathfrak{z}$, use $< \cdot, \cdot >$ as a generic symbol for the fixed inner products on $\mathfrak{v}, \mathfrak{z}$ and $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$, and we will contrast properties of objects arising from pairs j, j' of linear maps from $\mathfrak{z}$ to $\mathrm{so}(\mathfrak{v})$.

Notation 1.2. (i) The metric Lie algebra defined as above from the data $(\mathfrak{v}, \mathfrak{z}, j)$ will be denoted $\mathfrak{g}(j)$ and the corresponding simply-connected Lie group will be denoted $G(j)$. The Lie group $G(j)$ is endowed with the left-invariant Riemannian metric g determined by the inner product $< \cdot, \cdot >$ on $\mathfrak{g}(j)$.

(ii) Explicitly, $G(j)$ may be identified diffeomorphically (though not isometrically) with the Euclidean space $\mathfrak{v} \times \mathfrak{z}$ consisting of all pairs (x, z) with $x \in \mathfrak{v}, z \in \mathfrak{z}$. The group product on $G(j)$ is given by

$$(x, z)(x', z') = (x + x', z + z' + \frac{1}{2}[x, x']).$$

The Lie algebra element in $\mathfrak{g}(j)$ determined by $x \in \mathfrak{v}, z \in \mathfrak{z}$ will be denoted by $x + z$ with the diffeomorphism $\exp : \mathfrak{g}(j) \to G(j)$ thereby expressed by $\exp(x + z) = (x, z)$. The exponential map restricts to a linear isomorphism between $\mathfrak{z} \subset \mathfrak{g}(j)$ and the derived group $[G, G]$ of G.

(iii) Suppose $\mathcal{L}$ is a lattice of full rank in $\mathfrak{z}$, i.e., $\overline{\mathfrak{z}} = \mathfrak{z}/\mathcal{L}$ is a torus. Denote by $\overline{G(j)}$ the quotient of the Lie group $G(j)$ by the discrete central subgroup $\exp(\mathcal{L})$. Then $\overline{G(j)}$ is again a connected Lie group with Lie algebra $\mathfrak{g}(j)$. Diffeomorphically, $\overline{G(j)}$ may be identified with $\mathfrak{v} \times \overline{\mathfrak{z}}$, and the exponential map $\overline{\exp} : \mathfrak{g}(j) \to \overline{G(j)}$ is expressed by

$$\overline{\exp}(x + z) = (x, \overline{z}) \text{ for } x \in v, z \in \mathfrak{z}, \text{ and } \overline{z} = z + \mathcal{L} \in \overline{\mathfrak{z}}.$$

We assign to $\overline{G}(j)$ the unique left-invariant Riemannian metric determined by $< \cdot, \cdot >$. Thus the canonical projection from $G(j)$ to $\overline{G}(j)$ given by $(x, z) \to (x, \overline{z})$ is a Riemannian covering map as well as a Lie group homomorphism.

(iv) For $B = \{x \in \mathfrak{v} : \|x\| \leq 1\}$ the unit ball around 0 in $\mathfrak{v}$ and for $\mathcal{L}$ as in (iii), denote by $M(j)$ the subset $B \times \overline{\mathfrak{z}} = \overline{\exp}(B + \mathfrak{z})$ of $\overline{G(j)}$ equipped with the inherited Riemannian structure. $M(j)$ is thus a compact Riemannian submanifold of $\overline{G(j)}$ of full dimension with boundary diffeomorphic to $S \times \overline{\mathfrak{z}}$ for S the unit sphere around 0 in $\mathfrak{v}$. (Here we are using the identifications described in (iii). $M(j)$ of course depends on the choice of $\mathcal{L}$, but we view this choice as fixed.)

Definition 1.3. Let $\mathfrak{v}$ and $\mathfrak{z}$ be as above.

(i) A pair j, j' of linear maps from $\mathfrak{z}$ to $\mathrm{so}(\mathfrak{v})$ will be called *equivalent*, denoted $j \simeq j'$, if there exist orthogonal linear operators A on $\mathfrak{v}$ and C on $\mathfrak{z}$ such that

$$Aj(z)A^{-1} = j'(C(z))$$

for all $z \in \mathfrak{z}$.

(ii) We will say j is *isospectral* to j', denoted $j \sim j'$, if for each $z \in \mathfrak{z}$, the eigenvalue spectra (with multiplicities) of $j(z)$ and $j'(z)$ coincide, i.e., there exists an orthogonal linear operator A_z for which

$$A_z j(z) A_z^{-1} = j'(z).$$

Proposition 1.4. *Let $\mathfrak{v}$ and $\mathfrak{z}$ be finite dimensional real inner product spaces, j and j' linear injections from $\mathfrak{z}$ to $so(\mathfrak{v})$, and $\mathcal{L}$ a lattice of full rank in $\mathfrak{z}$. Let $\mathfrak{g}(j)$, $G(j)$, $\overline{G(j)}$, and $M(j)$ be the objects defined in 1.2 from the data $(\mathfrak{v}, \mathfrak{z}, j, \mathcal{L})$ and let $\mathfrak{g}(j')$, $G(j')$, $\overline{G(j')}$, and $M(j')$ be the corresponding objects defined by the data $(\mathfrak{v}, \mathfrak{z}, \ j', \mathcal{L})$. Then the following are equivalent:*

(a) $\overline{G(j)}$ is locally isometric to $\overline{G}(j')$;

(b) $M(j)$ is locally isometric to $M(j')$;

(c) $G(j)$ is isometric to $G(j')$;

(d) $j \simeq j'$ in the sense of Definition 1.3.

Proof. The local geometries of $G(j), \overline{G}(j)$, and $M(j)$ are identical. Thus each of (a) and (b) is equivalent to saying that $G(j)$ is locally isometric to $G(j')$ which, by simple-connectivity, is equivalent to (c). The second author showed in [27] that if (G, g) and (G', g') are two simply-connected nilpotent Lie groups with left-invariant metrics g, g' and associated metric Lie algebras $(\mathfrak{g} < \cdot, \cdot >)$, $(\mathfrak{g}', < \cdot, \cdot >')$, then (G, g) is isometric to (G', g'), if and only if there exist a map $\tau : \mathfrak{g} \to \mathfrak{g}'$ which is both a Lie algebra isomorphism and an inner product space isometry. In our case, equivalence of (c) and (d) follows by routine use of (1.1) serving to reduce these conditions on τ to $j \simeq j'$.

Theorem 1.5. *Let $\mathfrak{v}$ and $\mathfrak{z}$ be inner product spaces, $j, j' : \mathfrak{z} \to so(\mathfrak{v})$ linear injections, $\mathcal{L}$ a lattice of full rank in $\mathfrak{z}$, and $M(j)$ and $M(j')$ the* *manifolds defined in 1.2 from the data* $(\mathfrak{v}, \mathfrak{z}, j, \mathcal{L})$ *and* $(\mathfrak{v}, \mathfrak{z}, j', \mathcal{L})$*, respectively. Suppose* $j \sim j'$ *in the sense of Definition 1.3(ii). Then* $M(j)$ *is both Dirichlet and Neumann isospectral to* $M(j')$*.*

The proof is similar to the argument given in [11] for the construction of isospectral metrics on nilmanifolds (compact quotients of nilpotent Lie groups by discrete subgroups). Before giving the proof, we give a geometric interpretation of the condition $j \sim j'$ and establish some notation.

1.6. Remarks and Notation. Suppose $j \sim j'$.

(i.) If $\mathfrak{z}$ is one-dimensional, then $j \simeq j'$ in the notation of Definition 1.3, with C being the identity operator on $\mathfrak{z}$. Thus the isometry conditions of Proposition 1.4 hold with the isometry τ from $G(j)$ to $G(j')$ given by $\tau(x,z) = (A(x), z)$ with A as in 1.3(i). If $\mathcal{L}$ is any lattice in $\mathfrak{z}$, the translations of $G(j)$ and $G(j')$ by elements of $\mathcal{L}$ commute with τ, and thus τ induces global isometries between $\overline{G(j)}$ and $\overline{G(j')}$ and between $M(j)$ and $M(j')$.

(ii.) If $\mathfrak{z}$ is higher-dimensional, then $G(j)$ need not be isometric to $G(j')$, but the two manifolds admit many isometric quotients. More precisely, consider any co-dimension-one subgroup W of the derived group of $G(j)$. Such a subgroup corresponds under the exponential map to a co-dimension-one subspace of $\mathfrak{z}$, equivalently to the kernel of a nontrivial linear functional λ on $\mathfrak{z}$. Let $\mathfrak{z}_\lambda$ be the orthogonal complement of W in $\mathfrak{z}$. Then the two-step nilpotent Lie group $G_\lambda(j) := G(j)/W$ with the induced Riemannian metric is associated as in 1.2 with the data $(\mathfrak{v}, \mathfrak{z}_\lambda, j_{|\mathfrak{z}_\lambda})$. Observe that $j_{|\mathfrak{z}_\lambda} \sim j'_{|\mathfrak{z}_\lambda}$ since $j \sim j'$. Thus by (i) and the fact that $\mathfrak{z}_\lambda$ is one-dimensional, we see that $G_\lambda(j)$ is isometric to $G_\lambda(j')$.

(iii.) If $\mathcal{L}$ is a lattice in $\mathfrak{z}$ and if $\lambda \in \mathcal{L}^*$, i.e., λ is integer-valued on $\mathcal{L}$, then the projection from $\mathfrak{z}$ to $\mathfrak{z}_\lambda$ maps $\mathcal{L}$ to a lattice $\mathcal{L}_\lambda$ in $\mathfrak{z}_\lambda$. The associated quotients $\overline{G_\lambda(j)}$ and $\overline{G_\lambda(j')}$, defined as in 1.2, are isometric. Under the identifications in 1.2, the isometry Ψ_λ is given by $\Psi_\lambda(x, \bar{\mathfrak{z}}) = (A_\lambda(x), \bar{\mathfrak{z}})$, where $A_\lambda \in \mathrm{so}(\mathfrak{v})$ satisfies $j'(z) = A_\lambda j(z) A_\lambda^{-1}$ for $z \in \mathfrak{z}_\lambda$. This isometry restricts to an isometry between the compact submanifolds $M_\lambda(j)$ and $M_\lambda(j')$ of $\overline{G_\lambda(j)}$ and $\overline{G_\lambda(j')}$ defined as in 1.2(iv).

(iv.) We will say two vectors λ and μ in $\mathcal{L}^*$ are *equivalent*, denoted $\lambda \sim \mu$, if they have the same kernel. Denote the equivalence class of λ by $[\lambda]$ and denote the set of equivalence classes by $[\mathcal{L}^*]$. Observe that $G_\lambda(j)$, $\overline{G_\lambda(j)}$ and $M_\lambda(j)$ depend only on the equivalence class of λ.

Lemma 1.7. *Let* $\pi_\lambda : \overline{G(j)} \to \overline{G_\lambda(j)}$ *be the homomorphic projec-*

tion. Then the Laplacians Δ *of* $\overline{G(j)}$ *and* Δ_λ *of* $\overline{G_\lambda(j)}$ *satisfy*

$$\pi_\lambda^* \circ \Delta_\lambda = \Delta \circ \pi_\lambda^*.$$

Proof. It is well-known that the conclusion holds provided that the projection is a Riemannian submersion with totally geodesic fibers. The elementary proof that these conditions hold in our case is identical to the proof of Proposition 1.5 in [11].

Note that π_λ gives $\overline{G(j)}$ the structure of a principal torus bundle. Moreover, π_λ restricts to a Riemannian submersion from $M(j)$ to $M_\lambda(j)$ whose fibers are flat tori.

1.8 Remark. π_0 corresponds to the canonical projection $\mathfrak{v}\times\bar{\mathfrak{z}} \to \mathfrak{v}$ in the notation of 1.2. Moreover, $\overline{G_0(j)}$ is a Euclidean space isometric to $(\mathfrak{v}, <\cdot,\cdot>)$. The fiber torus is isometric to $(\bar{\mathfrak{z}}, <\cdot,\cdot>)$. In particular the fact that π_0 is a Riemannian submersion implies that the Riemannian measure on $\overline{G(j)}$ coincides with the Lebesgue measure on $\mathfrak{v}\times\bar{\mathfrak{z}}$.

Proof of Theorem 1.5. In the notation of 1.2, the derived group of the Lie group $\overline{G(j)}$ is identified with the torus $\bar{\mathfrak{z}}$. This torus acts isometrically on $\overline{G(j)}$ and on the submanifold $M(j)$ by left translations. The resulting action of $\bar{\mathfrak{z}}$ on $L^2(M(j))$, given by

$$(1) \qquad (\rho(\bar{w})f)(x,\bar{z}) = f(x, \bar{z}+\bar{w}),$$

clearly carries the space of smooth functions with Dirichlet boundary conditions to itself. To see that it also leaves invariant the space of smooth functions with Neumann boundary conditions, observe that the normal derivative of a function f across the boundary of $M(j)$ at the point $(x,\bar{z})$, where x is a unit vector in $\mathfrak{v}$, is given by $xf(x,\bar{z})$ where xf denotes the left-invariant vector field x on $\overline{G(j)}$ applied to f. Indeed

$$xf(x,\bar{z}) = \frac{d}{dt}f((x,\bar{z})\overline{exp}(tx)) = \frac{d}{dt}f((x,\bar{z})(tx,0)) = \frac{d}{dt}f((1+t)x,\bar{z})$$

by 1.2(ii),(iii). Since the torus $\bar{\mathfrak{z}}$ lies in the center of $\overline{G(j)}$, the torus action ρ commutes with all left-invariant vector fields. In view of the definition (1) of ρ, it follows that ρ leaves invariant the space of smooth functions with Neumann boundary conditions.

By Fourier decomposition on the torus, we can write

$$L^2(M(j)) = L^2(B\times\bar{\mathfrak{z}}) = \sum_{\lambda\in\mathcal{L}^*}\mathcal{H}_\lambda,$$

where

$$\mathcal{H}_\lambda = \{f \in L^2(B \times \bar{\mathfrak{z}}) : \rho(\bar{z})f = e^{2\pi i\lambda(z)}f \text{ for all } \bar{z} \in \bar{\mathfrak{z}}\}.$$

By the comments above, the space of smooth functions on $M(j)$ with Dirichlet, respectively Neumann, boundary conditions decomposes into its intersections with the $\mathcal{H}_\lambda$. To avoid cumbersome notation, we will refer to $\mathrm{spec}(\Delta_{|\mathcal{H}_\lambda})$ with Dirichlet (or Neumann) boundary conditions to mean the spectrum of the Laplacian of $M(j)$ restricted to the space of smooth functions in $\mathcal{H}_\lambda$ with the appropriate boundary conditions.

Set

$$\mathcal{H}_{[\lambda]} = \sum_{\mu\sim\lambda} \mathcal{H}_\mu.$$

(See Notation 1.6(iv).) Define $\mathcal{H}'_\lambda$ and $\mathcal{H}'_{[\lambda]}$ similarly using the data $(\mathfrak{v}, \mathfrak{z}, j', \mathcal{L})$.

By Lemma 1.7 and Remark 1.8, π_0^* intertwines the Laplacian Δ of $M(j)$, restricted to $\mathcal{H}_0$, with the Euclidean Laplacian on the ball B and similarly for the Laplacian Δ' of $M(j')$, restricted to $\mathcal{H}'_0$. Thus with either Dirichlet or Neumann boundary conditions, we have

$$\mathrm{spec}(\Delta_{|\mathcal{H}_0}) = \mathrm{spec}(\Delta'_{|\mathcal{H}'_0}). \tag{2}$$

Next for $0 \neq \lambda \in \mathcal{L}^*$, the map π_λ^* is a unitary map from $L^2(M_\lambda(j))$ to $\mathcal{H}_0 \oplus \mathcal{H}_{[\lambda]}$ (i.e., to $\{f \in L^2(B \times \bar{\mathfrak{z}}) : \rho(\bar{z})f = f \text{ for all } z \in \ker(\lambda)\}$). Thus by 1.6(iii) and Lemma 1.7, we have with either Dirichlet or Neumann boundary conditions that $\mathrm{spec}(\Delta_{|\mathcal{H}_0 \oplus \mathcal{H}_{[\lambda]}}) = \mathrm{spec}(\Delta'_{|\mathcal{H}'_0 \oplus \mathcal{H}'_{[\lambda]}})$. In view of equation (2), we thus have with either boundary condition that $\mathrm{spec}(\Delta_{|\mathcal{H}_{[\lambda]}}) = \mathrm{spec}(\Delta'_{\mathcal{H}'_{[\lambda]}})$ for every $\lambda \in \mathcal{L}^*$. The theorem now follows.

1.9 Remarks. (i) The intertwining operator T between the Laplacians of $M(j)$ and $M(j')$ can be written explicitly as $T = \oplus_{\lambda\in\mathcal{L}^*} T_\lambda$ where $T_\lambda : \mathcal{H}_\lambda \to \mathcal{H}'_\lambda$ is given by $(T_\lambda f)(x, \bar{z}) = f(A_\lambda^{-1}(x), \bar{z})$ with A_λ as in 1.6(iii).

(ii) By replacing the ball B with the vector space $\mathfrak{v}$ everywhere in the argument above, one obtains a unitary isomorphism $T : L^2(\overline{G(j)}) \to L^2(\overline{G(j')})$ satisfying $\Delta' = T \circ \Delta \circ T^{-1}$, where T is given by the same formula as in (i), but with $\mathcal{H}_\lambda$ now being a subspace of $L^2(\mathfrak{v} \times \bar{\mathfrak{z}})$.

(iii) By working with the Fourier transform on $L^2(\mathfrak{v} \times \mathfrak{z})$ with respect to the second variable, one can similarly obtain a unitary isomorphism between $L^2(G(j))$ and $L^2(G(j'))$ which intertwines the Laplacians. (There are some technical complications in the proof; for example, to define T, one needs $A_\lambda^{-1}(x)$ to be measurable as a function of

$(x, \lambda) \in \mathfrak{v} \times \mathfrak{z}^*$. Note that the A_z's in Definition 1.3(ii), and thus the A_λ's in Remark 1.6, are not uniquely determined. We have shown that one can choose the A_z's so that the map $z \to A_z$ from $\mathfrak{z}$ to the orthogonal group $O(\mathfrak{v})$ is in fact real analytic on a Zariski open subset of $\mathfrak{z}$.) Since $G(j)$ is diffeomorphic to $\mathbf{R}^n$ for some n, we thus obtain metrics on $\mathbf{R}^n$ whose Laplacians are intertwined. We omit the details here as we are currently investigating the behavior of the scattering operators for these metrics. We expect to address this issue in a later paper.

Example 1.10. In [10], [11], examples were given of pairs of isospectral (in the sense of Definition 1.3), inequivalent linear maps $j, j' : \mathfrak{z} \to \mathrm{so}(\mathfrak{v})$, where $\mathfrak{z}$ was 3-dimensional and $\mathfrak{v}$ was $4n$-dimensional with $n \geq 2$. The resulting isospectral manifolds, given by Theorem 1.5, thus have minimum dimension eleven. (The fact that j and j' give rise to isospectral compact manifolds with boundary was not observed in [10], [11]. Instead j and j' were used to construct isospectral closed manifolds using the method described in §3 below.) We now construct 7-dimensional examples. As we'll see in §4, these have quite different geometric properties from the earlier examples.

Let H be the quaternions and P the pure quaternions, i.e., $P = \{q \in H : \bar{q} = -q\}$. For $q \in H$, let $L(q)$ and $R(q)$ denote left and right multiplication by q on H. For $q, p \in P$, set $J(q,p) = L(q) + R(p)$. Then $J(q,p)$ is skew-symmetric relative to the standard inner product on H. Indeed the decomposition $\mathrm{so}(4) = \mathrm{so}(3) + \mathrm{so}(3)$ asserts that all skew-symmetric operators are of this form. An easy computation shows that the eigenvalues of $J(q,p)$ are $\pm i\sqrt{|q|^2 + |p|^2}$ and $\pm i\sqrt{|(|q|^2 - |p|^2)|}$; in particular, the spectrum of $J(q,p)$ depends only on the lengths of q and p.

Now let $\mathfrak{v} = H$, viewed as $\mathbf{R}^4$ with the standard inner product, and let $\mathfrak{z} = P$, viewed as $\mathbf{R}^3$ with the standard inner product. Let T and T' be fixed invertible linear operators on P such that $T' = A \circ T$ where A is an orthogonal operator of determinant -1. Define $j, j' : \mathfrak{z} \to \mathrm{so}(\mathfrak{v})$ by $j(q) = J(q, Tq)$ and $j'(q) = J(q, T'q)$ for all $q \in P$. Then $j \sim j'$ in the sense of Definition 1.3.

We next check whether j is equivalent to j'. The group $\mathrm{SO}(\mathfrak{v})$ consists of all operators $L(a)R(b)$ where a and b are unit quaternions. Conjugation of $J(q,p)$ by $L_a R_b$ gives $J(a^{-1}qa, bpb^{-1})$. All orthogonal transformations of $\mathfrak{v}$ are compositions of elements of $\mathrm{SO}(\mathfrak{v})$ with the quaternionic conjugation map B of $\mathfrak{v}$. Conjugation of $J(q,p)$ by B yields $J(-p,-q)$. Since $\det T' = -\det(T)$, it follows easily that the construction above always yields inequivalent maps j and j'.

With any choice of lattice $\mathcal{L}$ in $\mathfrak{z}$, Theorem 1.5 yields pairs of isospectral 7-dimensional compact manifolds with boundary which are not locally isometric.

2. Examples of isospectral Lie algebra deformations

Definition 2.1. Let $\mathfrak{v}$ and $\mathfrak{z}$ be finite dimensional inner product spaces and j_0 any linear map from $\mathfrak{z}$ to $\mathrm{so}(\mathfrak{v})$. By a *d-parameter non-trivial isospectral deformation* of j_0 we mean a continuous function $u \mapsto j_u$ from a pathwise connected subset D of $\mathbf{R}^d$ having non-empty interior into the space of linear maps from $\mathfrak{z}$ to $\mathrm{so}(\mathfrak{v})$ such that

(i) $j_0 = j_{u_0}$ for some $u_0 \in D$;

(ii) $j_u \sim j_0$ for all $u \in D$ (see Definition 1.3(ii));

(iii) $j_u \not\simeq j_{u'}$ whenever u and u' are distinct points in D (see Definition 1.3(i)).

Equivalently, $\mathcal{G} = \{\mathfrak{g}(j_u) : u \in D\}$ is a family containing $\mathfrak{g}(j_0)$ of nilpotent metric Lie algebras all having $\mathfrak{v} \oplus \mathfrak{z}$ as their underlying vector space, and the structure constants of $\mathfrak{g}(j_u)$ relative to any fixed bases of $\mathfrak{v}$ and $\mathfrak{z}$ depend continuously on the parameter d-tuple u. Any choice of lattice $\mathcal{L}$ of maximal rank in $\mathfrak{z}$ gives rise to a d-parameter family $\{M(j_u)\}_{u\in D}$ of isospectral compact manifolds with boundary as in Theorem 1.5.

Throughout this section, we will consider the special case where $\dim \mathfrak{z} = 2$ with $m = \dim \mathfrak{v}$ variable. Our goal is to show that when either $m = 5$ or $m \geq 7$, every "generic" j_0 admits a d-parameter non-trivial isospectral deformation with $d > 1$. For $m = 6$, we will exhibit explicitly one-parameter deformations for certain j_0 of a restrictive type. For $m \leq 4$, straightforward calculations show that any two isospectral j's are in fact equivalent, so non-trivial isospectral deformations of this type are impossible.

Theorem 2.2. *Let* $\dim \mathfrak{z} = 2$*, let* $m = \dim \mathfrak{v}$ *be any positive integer other than* $1, 2, 3, 4,$ *or* 6*, and let* L *be the real vector space consisting of all linear maps from* $\mathfrak{z}$ *to* $so(\mathfrak{v})$*. Then there is a Zariski open subset* $\mathcal{O}$ *of* L *(i.e.,* $\mathcal{O}$ *is the complement of the set of roots of some non-zero* *polynomial function on L) such that each $j_0 \in \mathcal{O}$ admits a d-parameter non-trivial isospectral deformation where*

$$d \geq m(m-1)/2 - [m/2]([m/2]+2) > 1.$$

Proof. For $j \in L$, let

$$I_j = \{j' \in L : j' \sim j\}, \qquad E_j = \{j' \in I_j : j' \simeq j\}. \tag{1}$$

The idea of the proof is to define $\mathcal{O}$ in such a way that for $j_0 \in \mathcal{O}$, $P_{j_0} := I_{j_0} \cap \mathcal{O}$ is an embedded submanifold of L, which can be foliated by its intersection with the sets $E_j, j \in P_{j_0}$, and for which there is a submanifold N_{j_0} of P_{j_0} transverse to the foliation. Any parametrization of N_{j_0} then defines a non-trivial isospectral deformation of j_0.

Let $l = [m/2]$ and, for $1 \leq k \leq l$, define $T_k : \mathrm{so}(\mathfrak{v}) \to \mathbf{R}$ by $T_k(C) = \mathrm{trace}(C^{2k})$. If C and C' are similar, i.e., have the same eigenvalues, then trivially $T_k(C) = T_k(C')$ for all k. But the converse is also true as can be seen by a standard combinatoric argument showing that the coefficients of powers of λ in the characteristic polynomial $\mathcal{X}(\lambda, C) = \det(\lambda Id - C)$ are polynomials in $\{T_1(C), \ldots, T_l(C)\}$. If we define $T_k : \mathfrak{z} \times L \to \mathbf{R}$ by $T_k(z, j) = T_k(j(z))$, this means that $j \sim j' \Leftrightarrow T_k(z, j) = T_k(z, j')$ for all $z \in \mathfrak{z}$ and all $k, 1 \leq k \leq l$. Moreover, each of the functions T_k is a polynomial on $\mathfrak{z} \times L$, which is separately homogeneous of degree $2k$ in each variable. If we fix any orthornormal basis $\{\epsilon_1, \epsilon_2\}$ of $\mathfrak{z}$ and denote a typical element $z \in \mathfrak{z}$ by $z = s\epsilon_1 + t\epsilon_2$, then expansion of $T_k(z, j) = \mathrm{trace}(sj(\epsilon_1) + tj(\epsilon_2))^{2k}$ into (s, t) monomials gives us $2k + 1$ coefficient functions which are polynomials in $j(\epsilon_1)$ and $j(\epsilon_2)$ and thus polynomials on L. Since $\sum_{k=1}^{l}(2k+1) = l(l+2)$, we conclude that there is a map $F : L \to \mathbf{R}^{l(l+2)}$ each of whose entries is a polynomial on L and for which $j \sim j' \Leftrightarrow F(j) = F(j')$. Let R be the maximum rank of F. Then R is the largest integer for which there is some $j \in L$ such that the tangent map $F_{*_j} : L \to \mathbf{R}^{l(l+2)}$ has rank R. Since each of the entries in any matrix representation of F_{*_j} is a polynomial in j and since a matrix has rank $\geq R$ precisely when the sum of the squares of the determinants of its $R \times R$ minors is non-zero, it follows that the subset $\mathcal{O}_1$ of L on which F has rank R is a Zariski open set. Moreover, for $j_0 \in \mathcal{O}_1$, the Implicit Function Theorem implies that there is a neighborhood $\mathcal{U}$ of j_0 in L for which $I_{j_0} \cap U = F^{-1}(F(j_0)) \cap U$ is an embedded submanifold of co-dimension R.

We now turn toward examination of the sets E_j in (1). The group $G = O(\mathfrak{v}) \times O(\mathfrak{z})$ acts on L by

$$((A, C) \cdot j)(z) = Aj(C^{-1}z)A^{-1}, \tag{2}$$

and, by Definition 1.3, $j' \simeq j \Leftrightarrow j' = (A, C) \cdot j$ for some $(A, C) \in G$. Let $1_{\mathfrak{v}}$ and $1_{\mathfrak{z}}$ denote the identity operators on $\mathfrak{v}$ and $\mathfrak{z}$. We now claim that there is a Zariski open subset $\mathcal{O}_2$ of L such that for each $j \in \mathcal{O}_2, E_j$ is the orbit of j under the subgroup $K := O(\mathfrak{v}) \times \{\pm 1_{\mathfrak{z}}\}$ and such that the stability subgroup of K at j is $\{(\pm 1_{\mathfrak{v}}, 1_{\mathfrak{z}})\}$. To see this, first consider any $j \in L$ and $(A, C) \in G$ such that $(A, C) \cdot j \in E_j$. Since $E_j \subset I_j$ and since $j \circ C^{-1} = (A^{-1}, 1) \cdot (A, C) \cdot j$, we see that $j \circ C^{-1} \in I_j$. Thus $T_k(z, j \circ C^{-1}) = T_k(z, j)$ for all z and k. In particular, C is orthogonal both with respect to the given inner product on $\mathfrak{z}$ and the quadratic form $z \mapsto T_1(z, j) = \mathrm{trace}(j(z))^2 = - < j(z), j(z) >$, where $< c, d >= \mathrm{trace}(c\, d^t) = -\mathrm{trace}(cd)$ is the standard inner product on $\mathrm{so}(\mathfrak{v})$. Relative to any orthonormal basis $\{\epsilon_1, \epsilon_2\}$ of $\mathfrak{z}, T_1(\cdot, j)$ has matrix

$$- \begin{bmatrix} |j(\epsilon_1)|^2 & < j(\epsilon_1), j(\epsilon_2) > \\ < j(\epsilon_1), j(\epsilon_2) > & |j(\epsilon_2)|^2 \end{bmatrix}.$$

Unless this matrix is a scalar multiple of the 2×2 identity matrix, i.e., unless

$$\phi_1(j) := (|j(\epsilon_1)|^2 - |j(\epsilon_2)|^2)^2 + < j(\epsilon_1), j(\epsilon_2) >^2$$

vanishes, there are precisely four transformations orthogonal with respect to both forms, namely $\pm 1_{\mathfrak{z}}$ and $\pm C_0$ where C_0 is the reflection leaving one eigenvector of the above matrix fixed while changing the sign of the other. For j not a root of the polynomial ϕ_1, we conclude that C is one of these four transformations. But $m \geq 5$ means $l = [m/2] \geq 2$ so C must also satisfy $T_2(z, j \circ C^{-1}) = T_2(z, j)$, i.e., $\mathrm{trace}(j(z))^4 = \mathrm{trace}(j(C^{-1}z))^4$ for all $z \in \mathfrak{z}$. By straightforward but tedious calculations, one can check that there is a fourth order polynomial ϕ_2 on L for which $T_2(\cdot, j \circ C_0^{-1}) \neq T_2(\cdot, j)$ when $\phi_2(j) \neq 0$. Thus when both $\phi_1(j)$ and $\phi_2(j)$ are non-zero, $(A, C) \cdot j \in E_j \Leftrightarrow (A, C) = (A, \pm 1_{\mathfrak{z}}) \in K$. In this case, $(A, C) \cdot j = j$ if and only if either $C = 1_{\mathfrak{z}}$ and $A \in O(\mathfrak{v})$ commutes with $j(z)$ for all z, or else $C = -1_{\mathfrak{z}}$ and A anti-commutes with $j(z)$ for all z. With $\{\epsilon_1, \epsilon_2\}$ as above and $j_1 = j(\epsilon_1), j_2 = j(\epsilon_2)$, it's easy to select choices of j_1 and j_2 for which no non-zero linear operator A on $\mathfrak{v}$ anti-commutes with both j_1 and j_2, while $\pm 1_{\mathfrak{v}}$ are the only orthogonal operators commuting with both j_1 and j_2. Moreover, these properties are equivalent to saying that the linear map $\phi_j(B) := (j_1B - Bj_1, j_2B - Bj_2)$ from $\mathfrak{gl}(\mathfrak{v})$ to $\mathfrak{gl}(\mathfrak{v}) \times \mathfrak{gl}(\mathfrak{v})$ has one-dimensional kernel while $\widetilde{\phi}_j(B) := (j_1B + Bj_1, j_2B + Bj_2)$ is injective, conditions which can be expressed by the statement that certain non-vanishing polynomials ϕ and $\widetilde{\phi}$ on L do not have j as a root. Combining all of these arguments, when j belongs to the complement $\mathcal{O}_2$ of the set of roots of $\phi_1^2 + \phi_2^2 + \phi^2 + \widetilde{\phi}^2$, the properties announced above in our claim are satisfied.

Let $\mathcal{O} = \mathcal{O}_1 \cap \mathcal{O}_2, j_0 \in \mathcal{O}$, and $P_{j_0} = I_{j_0} \wedge \mathcal{O}$. From above, P_{j_0} is a smooth manifold whose dimension is $\dim L - R \geq m(m-1) - [m/2]([m/2]+2)$. For $K = O(m) \times \{\pm 1_{\mathfrak{z}}\}$, it's trivial to check that when any one of the polynomials defining $\mathcal{O}$ does not vanish at j, the same is true for each member of the $m(m-1)/2$-dimensional orbit $K \cdot j$; i.e., $\mathcal{O}$ is closed under the action of K. Moreover, for $j \in \mathcal{O}$, we have shown that the orbit $K \cdot j$ coincides with E_j and the stability subgroup at j is $Z := \{(\pm 1_{\mathfrak{v}}, 1_{\mathfrak{z}})\}$. This means that the compact group K/Z acts freely on the manifold P_{j_0} with orbits expressing equivalence of elements. By the properties of compact transformation groups (e.g. [3, p. 82–86]), there is a submanifold N_{j_0} of P_{j_0} such that $(j, \tilde{K}) \mapsto \tilde{K}(j)$ is a homeomorphism from $N_{j_0} \times (K/Z)$ onto an open neighborhood of j_0 in P_{j_0}. The dimension of N_{j_0} is then

$$d = \dim I_{j_0} - m(m-1)/2 \geq \frac{m(m-1)}{2} - [m/2]([m/2]+2).$$

For $m = 5$ or $m \geq 7$, clearly $d > 1$ and any local parameterization of N_{j_0} defines a d-parameter non-trivial isospectral deformation of j_0.

2.3 Eight dimensional examples. For $m = 6$, the argument in the proof of Theorem 2.2 breaks down since $m(m-1)/2 - [m/2]([m/2]+2) = 15 - 15 = 0$. In the language of the proof of Theorem 2.2, the examples below correspond to choosing certain j_0's where the rank of the polynomial map F is less than R with the result being that the isospectral family I_{j_0} in equation (1) is four-dimensional, while the sets E_j contained in I_{j_0} are three-dimensional and admit a one-parameter transversal. Lengthy and non-illuminating calculations are avoided by fixing orthonormal bases for $\mathfrak{v}$ and $\mathfrak{z}$ and simply defining in concrete matrix terms the members of the transversal.

Thus, take $\mathfrak{z} = \mathbf{R}^2$ and $\mathfrak{v} = \mathbf{R}^6$ with their standard ordered bases and standard inner product. For $a, b \in \mathrm{so}(6)$ and $s, t \in \mathbf{R}$, define $j_{a,b}(s,t) = sa + tb$. Each linear map $j : \mathbf{R}^2 \to \mathrm{so}(6)$ is of the form $j = j_{a,b}$ for some $a, b \in \mathrm{so}(6)$. Fix for the remainder of the discussion an element $a \in \mathrm{so}(6)$ which is in block diagonal form with 2×2 diagonal blocks $a_i \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $1 \leq i \leq 3$, where $0 < a_1 < a_2 < a_3$. Consider all matrices $b \in \mathrm{so}(6)$ of the form

$$b = \begin{bmatrix} 0 & 0 & b_{12} & 0 & b_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -b_{12} & 0 & 0 & 0 & b_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -b_{13} & 0 & -b_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with $(b_{12}, b_{13}, b_{23}) \in \mathbf{R}^3 - \{0\}$.

We first note that if b and b' are of this form, then $j_{a,b} \simeq j_{a,b'} \Leftrightarrow b' = \pm b$. Indeed, in the notation of equation (2), if $j_{a,b'} = (A, C) \cdot j_{a,b}$ for some $A \in O(6), C \in O(2)$, then for $\epsilon_2 = (0,1), j_{a,b'}(\epsilon_2) = b'$ is a rank-2 matrix similar to $j_{a,b}(C^{-1}\epsilon_2)$. But a simple calculation shows that $j_{a,b}(s,t)$ has rank 2 only when $s = 0$. It follows that $C\epsilon_2 = \pm\epsilon_2$, so C is one of $\pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and then $AaA^{-1} = \pm a$, $AbA^{-1} = \pm b'$. Since a_1, a_2, a_3 are distinct, this forces A to be in block diagonal form with 2×2 diagonal blocks which either all commute or all anticommute with $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Using the specific form of b and b', it follows in either case that $AbA^{-1} = b$ so $b' = \pm b$.

Next an easy direct calculation yields

$$\det\{\lambda Id - j_{a,b}(s,t)\} = \prod_{i=1}^{3}(\lambda^2 + s^2 a_i^2) + \lambda^4 t^2 \sum_{i<j} b_{ij}^2 + \lambda^2 s^2 t^2 (a_1^2 b_{23}^2 + a_2^2 b_{13}^2 + a_3^2 b_{12}^2).$$

Comparing coefficients, it follows that $j_{a,b} \sim j_{a,b'} \Leftrightarrow (b_{12}, b_{13}, b_{23})$ and $(b'_{12}, b'_{13}, b'_{23})$ satisfy the equations

$$(i) \qquad \sum_{i<j} b_{ij}^2 - (b'_{ij})^2 = 0$$

and

$$(ii) \qquad a_1^2(b_{23}^2 - (b'_{23})^2) + a_2^2(b_{13}^2 - (b'_{13})^2) + a_3^2(b_{12}^2 - (b'_{12})^2) = 0.$$

In view of equation (i), equation (ii) can be rewritten as

$$(ii') \qquad \sum_{i<j} (a_i^2 + a_j^2)(b_{ij}^2 - (b'_{ij})^2) = 0.$$

The general solution of equations (i) and (ii') is

$$(*) \qquad \begin{aligned} (b'_{12})^2 &= b_{12}^2 + u(a_2^2 - a_1^2), \\ (b'_{13})^2 &= b_{13}^2 + u(a_1^2 - a_3^2), \\ (b'_{23})^2 &= b_{23}^2 + u(a_3^2 - a_2^2), \end{aligned}$$

where u is any real number in the closed interval

$$I = \left[\max\left(\frac{-b_{12}^2}{a_2^2 - a_1^2}, \frac{-b_{23}^2}{a_3^2 - a_2^2}\right), \frac{b_{13}^2}{a_3^2 - a_1^2}\right].$$

If we take any b for which I has non-empty interior and, for each $u \in I$, define $b(u)$ as the unique solution of the above equations for which $b_{ij}(u)$ has the same sign as b_{ij} for all i, j, it follows that $u \to j_{a,b(u)}$ is a 1-parameter non-trivial isospectral deformation of $j_{a,b}$.

3. Compact nilmanifolds

A compact Riemannian nilmanifold is a quotient $N = \Gamma\backslash G$ of a simply-connected nilpotent Lie group G by a (possibly trivial) discrete subgroup Γ, together with a Riemannian metric g whose lift to G is left-invariant.

We now recall a method, developed in [11], for constructing isospectral compact Riemannian nilmanifolds. For convenience, we'll restrict our attention to two-step nilmanifolds, although Theorem 3.2 below can be formulated in the context of nilmanifolds of arbitrary step. Even in the two-step case, the formulation of Theorem 3.2 in [11, Theorem 1.8] is slightly more general than that given here.

Notation and Remarks 3.1. (i) A nilpotent Lie group G admits a co-compact discrete subgroup Γ if and only if the Lie algebra $\mathfrak{g}$ of G has a basis $\mathcal{B}$ relative to which the constants of structure are integers (see [22]). If $\mathcal{B}$ is such a basis and $\mathcal{A}$ is the integer span of $\mathcal{B}$, then $\exp(\mathcal{A})$ generates a co-compact discrete subgroup of G. Conversely, if Γ is a co-compact discrete subgroup of G, then $\log(\Gamma)$ spans a lattice of full rank in $\mathfrak{g}$, where $\log : G \to \mathfrak{g}$ is the inverse of the Lie group exponential map.

(ii) We use the notation of 1.1 and 1.2. Thus a simply-connected nilpotent Lie group $G = G(j)$ with a left-invariant metric is defined by data $(\mathfrak{v}, \mathfrak{z}, j)$. If Γ is a co-compact discrete subgroup of G, then Γ intersects $[G, G]$ in a lattice of full rank $\mathcal{L}$, which we may also view as a lattice in $\mathfrak{z}$ under the identification in 1.2. In summary, a compact nilmanifold $N = \Gamma\backslash G$ is defined by the data $(\mathfrak{v}, \mathfrak{z}, j, \Gamma)$ and Γ determines a lattice $\mathcal{L}$ in $\mathfrak{z}$. In the sequel, we will consider fixed $(\mathfrak{v}, \mathfrak{z}, \mathcal{L})$ but vary the choice of j with the requirement that the resulting simply-connected nilpotent Lie group $G(j)$ admit a co-compact discrete subgroup Γ whose intersection with the derived group of $G(j)$ is given by $\mathcal{L}$. We will denote the nilmanifold $\Gamma\backslash G(j)$ by $N(j, \Gamma)$.

(iii) We continue to use the notation of 1.6 as well. For $\lambda \in \mathcal{L}^*$, the projection $G(j) \to G_\lambda(j) = G(j)/\ker(\lambda)$ sends Γ to a co-compact discrete subgroup Γ_λ. We denote by $N_\lambda(j, \Gamma)$ the quotient $\Gamma_\lambda\backslash G_\lambda(j)$ with the Riemannian metric induced by that of $G_\lambda(j)$. Note that $N_0(j, \Gamma)$ is a flat torus. Letting $\mathcal{A}_\mathfrak{v}$ be the image of $\log(\Gamma)$ under the orthogonal projection from $\mathfrak{g}(j)$ to $\mathfrak{v}$, then $N_0(j, \Gamma)$ is isometric to the torus $\mathfrak{v}/\mathcal{A}_\mathfrak{v}$ with the flat metric defined by the inner product on $\mathfrak{v}$.

Theorem 3.2. [11] *Let $N(j, \Gamma)$ and $N(j', \Gamma')$ be compact Riemannian nilmanifolds associated with the data $(\mathfrak{v}, \mathfrak{z}, \mathcal{L})$ as in 3.1. Suppose that $spec(N_\lambda(j, \Gamma)) = spec(N_\lambda(j', \Gamma'))$ for every $\lambda \in \mathcal{L}^*$. Then $spec(N(j, \Gamma)) = spec(N(j', \Gamma'))$.*

We wish to correct an error in the version of this theorem given in [11], Theorem 1.8: One must assume that the correspondence $\lambda \to \lambda'$ given there is norm-preserving if $\dim(\mathfrak{z}) - 1$. This assumption is actually satisfied in all the applications of Theorem 1.8 given in [11].

Definition 3.3. We will say a two-step nilpotent Lie group $G = G(j)$ is *non-singular* if $j(z)$ is non-singular for all $z \in \mathfrak{z}$. We will also say any associated compact nilmanifold $N(j, \Gamma)$ is non-singular in this case.

In [11], we studied non-singular nilmanifolds and proved the following as a consequence of Theorem 3.2:

Theorem 3.4. *In the notation of 3.1, let $N(j, \Gamma)$ and $N(j', \Gamma')$ be compact non-singular two-step Riemannian nilmanifolds associated with the same data $(\mathfrak{v}, \mathfrak{z}, \mathcal{L})$. Assume the following:*

(i) $spec(N_0(j, \Gamma)) = spec(N_0(j', \Gamma'))$ and

(ii) $j \sim j'$. (See Definition 1.3(ii).)

Then $spec(N(j,\Gamma)) = spec(N(j',\Gamma'))$.

Example 3.5. Examples of minimum dimension 11 were given in [10], [11]. We now construct compact quotients of the pairs of 7-dimensional isospectral simply-connected manifolds $G(j)$ and $G(j')$ constructed in Example 1.10.

In the notation of Example 1.10, observe that the constants of structure of $\mathfrak{g}(j)$ relative to the "standard" basis are integers provided that the matrix entries of T relative to the standard basis of $\mathfrak{z}$ are integers. Thus we assume that the matrix entries of both T and T' are integers. We can then, for example, let $\mathcal{A}$ and $\mathcal{A}'$ be the integer span of the standard basis elements of $\mathfrak{v}$ and $\mathfrak{z}$, and let Γ and Γ' be the discrete subgroups of $G(j)$ and $G(j')$ generated by $\exp(\mathcal{A})$ and $\exp'(\mathcal{A}')$, respectively. (See 3.1(i).) The nilmanifolds $N(j,\Gamma)$ and $N(j',\Gamma')$ trivially satisfy condition (i) of Theorem 3.4; in fact, the tori $N_0(j,\Gamma)$ and $N_0(j',\Gamma')$ are isometric. Moreover, condition (ii) of Theorem 3.4 is automatic from the construction in 1.10. Thus the nilmanifolds $N(j,\Gamma)$ and $N(j',\Gamma')$ are isospectral.

The non-singular compact nilmanifolds N are particularly easy to work with as the quotient manifolds $N_\lambda(j,\Gamma)$ defined in 3.1(iii) are Heisenberg manifolds when $\lambda \neq 0$; that is, the center of G_λ is one-dimensional. In [14], the authors gave sufficient conditions for two Heisenberg manifolds to be isospectral. (Pesce [21] later proved these conditions are also necessary.) These conditions are used in [11] to prove Theorem 3.4.

We want to find isospectral compact quotients of some pairs of simply-connected nilpotent Lie groups associated with the Lie algebras constructed in §2. Thus we need to generalize Theorem 3.4 to the possibly singular case. As always, we will assume that $j(z)$ is non-zero for all $z \in \mathfrak{z}$ (i.e., that $\mathfrak{z} = [\mathfrak{g}(j), \mathfrak{g}(j)]$). When $\lambda \neq 0$, the quotient $\mathfrak{g}_\lambda(j)$ has one-dimensional derived algebra but may have a higher-dimensional center. The corresponding Lie group is of the form $G_\lambda(j) = H \times A$, where H is a Heisenberg group and A an abelian group. Thus in view of Theorem 3.2, we first need to examine isospectrality conditions for compact quotients of groups of this form.

Notation 3.6. In the notation of 3.1, consider a nilmanifold $N(j,\Gamma)$ with $\mathfrak{z}$ one-dimensional. We can write $\mathfrak{v}$ as an orthogonal direct sum $\mathfrak{v} = \mathfrak{u} \oplus \mathfrak{a}$ where $\mathfrak{a} = \ker(j(z))$ for $0 \neq z \in \mathfrak{z}$. (Note that $\mathfrak{a}$ is independent of the choice of z since $\mathfrak{z}$ is one-dimensional.) The Lie algebra $\mathfrak{g}(j)$ then splits into an orthogonal sum of ideals $\mathfrak{h} \oplus \mathfrak{a}$, where $\mathfrak{h} = \mathfrak{u} + \mathfrak{z}$ is a Heisenberg algebra.

Since $\mathfrak{a} + \mathfrak{z}$ is the center of $\mathfrak{g}(j)$, $\log(\Gamma)$ intersects $\mathfrak{a} + \mathfrak{z}$ in a lattice $\mathcal{K}$ of maximal rank. (See [22].) Let $\mathcal{K}^*$ denote the dual lattice in $(\mathfrak{a} + \mathfrak{z})^*$. The inner product $<, >$ on $\mathfrak{a} + \mathfrak{z}$ defines a dual inner product on $(\mathfrak{a} + \mathfrak{z})^*$ and thus defines a norm $\| \ \|$ on $\mathcal{K}^*$.

Proposition 3.7. *Using the notation of 3.1 and 3.6, let $N(j, \Gamma)$ be a compact nilmanifold and assume $\mathfrak{z}$ is one-dimensional. Then spec$(N(j, \Gamma))$ is completely determined by the following data:*

(i) spec$(N_0(j, \Gamma))$.

(ii) The eigenvalues of the linear operator $j(z)$, where z is a unit vector in $\mathfrak{z}$. (Since $\mathfrak{z}$ is one-dimensional and $j(z)$ is skew, the eigenvalues of $j(z)$ are independent of the choice of unit vector z.)

(iii) $\{(\sigma(z), \|\sigma\|) \in \mathbf{R}^2 : \sigma \in \mathcal{K}^*\}$ where z is given as in (ii).

The case in which M is a Heisenberg manifold is proven in [14] and is the key lemma used in Theorem 3.4 above. Proposition 3.7 will be proved in the Appendix.

3.8 Remark. In the special case that $\mathcal{K} = (\mathcal{K} \cap \mathfrak{z}) \oplus (\mathcal{K} \cap \mathfrak{a})$, the data (iii) can be expressed more simply. The inner product $<, >$ defines flat Riemannian metrics on the circle $\mathfrak{z}/(\mathcal{K} \cap \mathfrak{z})$ and the torus $\mathfrak{a}/(\mathfrak{a} \cap \mathcal{K})$. Specifying the data (iii) is equivalent to specifying the length of this circle and the spectrum of this torus.

Notation and Remarks 3.9. In 2.3, we considered a class of eight-dimensional metric Lie algebras $\mathfrak{g}(j_{a,b})$. We now show that for certain choices of pairs $j = j_{a,b}$ and $j' = j_{a,b'}$, the associated nilpotent Lie groups $G(j)$ and $G(j')$ admit isospectral compact quotients. First observe that if the matrix entries a_1, a_2, a_3 of a and b_{12}, b_{13}, b_{23} of b are integers, then the constants of structure of $\mathfrak{g}(j_{a,b})$ with respect to the standard bases $\{e_1, \ldots, e_6\}$ of $\mathfrak{v} = \mathbf{R}^6$ and $\{\epsilon_1, \epsilon_2\}$ of $\mathfrak{z} = \mathbf{R}^2$ are integers. Thus, if we let $\mathcal{A}$ be the lattice in $\mathfrak{v} + \mathfrak{z}$ spanned by $\{e_1, \ldots, e_6, \epsilon_1, \epsilon_2\}$, then $\exp(\mathcal{A})$ generates a co-compact discrete subgroup $\Gamma_{a,b}$ of $G(j_{a,b})$. (See 3.1.)

Theorem 3.10. *In the notation of 2.3 and 3.9, assume that the matrix entries of a, b, and b' are integers, that g.c.d.(b_{12}, b_{13}, b_{23}) = g.c.d.$(b'_{12}, b'_{13}, b'_{23})$, and that the condition (*) of 2.3 is satisfied. Then* *the compact Riemannian nilmanifolds* $N(j_{a,b},\Gamma_{a,b})$ *and* $N(j_{a,b'},\Gamma_{a,b'})$ *are isospectral.*

Proof. Write $N = N(j_{a,b},\Gamma_{a,b})$ and $N' = N(j_{a,b'},\Gamma_{a,b'})$. We apply Theorem 3.2. In the notation of 3.1(ii), the lattice in $\mathfrak{z} = \mathbf{R}^2$ associated with both $\Gamma_{a,b}$ and $\Gamma_{a,b'}$ is given by $\mathcal{L} = \mathrm{span}_{\mathbf{Z}}\{\epsilon_1,\epsilon_2\} = \mathbf{Z}^2$.

Let $\lambda \in \mathcal{L}^*$. In case $\lambda = 0$, both N_0 and N_0' are isometric to the 6-dimensional cubical torus, so $\mathrm{spec}(N_0) = \mathrm{spec}(N_0')$ holds trivially. Next, observe that $j(s\epsilon_1 + t\epsilon_2) = sa + tb$ is non-singular except when $s = 0$. Thus, if $\lambda(\epsilon_1) \neq 0$, then $j_{a,b}(z)$ and $j_{a,b'}(z)$ are non-singular similar operators for all $z \in \mathfrak{z}_\lambda$, and therefore N_λ and N_λ' are Heisenberg manifolds. Proposition 3.7 (see the simplified version 3.8 with $\mathfrak{a} = 0$) implies $\mathrm{spec}\,(N_\lambda) = \mathrm{spec}\,(N_\lambda')$.

It remains to consider the case $\ker(\lambda) = \mathbf{R}\epsilon_1$. In this case, $G_\lambda(j_{a,b})$ and $G_\lambda(j_{a,b'})$ are isomorphic as Lie groups to $H \times A$, where H is the 3-dimensional Heisenberg group and $A = \mathbf{R}^4$. Letting $\pi_\lambda : G(j_{a,b}) \to G_\lambda(j_{a,b})$ be the projection and writing $\overline{X} = \pi_{\lambda *}(X)$ for X in the Lie algebra $\mathfrak{g}(j_{a,b})$, we have in the notation of 1.6, 3.6, and 3.9 that

$$\begin{aligned}
\mathfrak{a} &= \pi_{\lambda_*}(\ker(b)) = \mathrm{span}\{\overline{e}_2,\overline{e}_4,\overline{e}_6, b_{23}\overline{e}_1 - b_{13}\overline{e}_3 + b_{12}\overline{e}_5\},\\
\mathfrak{u} &= \pi_{\lambda_*}(\mathfrak{v} \ominus \ker(b)),
\end{aligned}$$

and $\mathfrak{z}_\lambda = \mathbf{R}\overline{\epsilon}_2$. Moreover, letting $\mathcal{K} = \pi_{\lambda *}(\mathcal{A}) \cap (\mathfrak{a} + \mathfrak{z}_\lambda)$, we have

$$\mathcal{K} = (\mathcal{K} \cap \mathfrak{z}_\lambda) \oplus (\mathcal{K} \cap \mathfrak{a})$$

with $\mathcal{K} \cap \mathfrak{z}_\lambda = \mathbf{Z}\overline{\epsilon}_2$ and $\mathcal{K} \cap \mathfrak{a} = \mathrm{span}_{\mathbf{Z}}\{\overline{e}_2,\overline{e}_4,\overline{e}_6,\overline{w}\}$ where

$$\overline{w} = \text{g.c.d.}(b_{12},b_{13},b_{23})(b_{23}\overline{e}_1 - b_{13}\overline{e}_3 + b_{12}\overline{e}_5).$$

Thus $\mathcal{K}$ is an orthogonal lattice isomorphic to $\mathbf{Z}^4 \times |\overline{w}|\mathbf{Z}$.

The analogous statements hold of course when b is replaced by b'. The data (ii) in Proposition 3.7 agree for N_λ and N_λ'; both are given by the eigenvalues of b. (Recall that b and b' are similar.) To see that the data (iii), as simplified in Remark 3.8, agree for N_λ and N_λ', we need only show that $|\overline{w}| = |\overline{w}'|$. This equality follows from the hypothesis of the theorem and the fact that $b_{12}^2 + b_{13}^2 + b_{23}^2 = (b_{12}')^2 + (b_{13}')^2 + (b_{23}')^2$, as can be seen from the isospectrality condition $(*)$ of 2.3.

Example 3.11. Fix a choice of a with integer entries a_1, a_2, a_3. It is easy to find pairs b and b' with integer entries b_{ij} and $b_{ij}', 1 \leq i < j \leq 3$, so that the isospectrality condition $(*)$ in 2.3 holds, i.e., so that $j_{a,b} \sim j_{a,b'}$. We need only choose the parameter u in $(*)$ so that each of $u(a_i^2 - a_j^2)$ is a difference of two squares; i.e, each $u(a_i^2 - a_j^2)$ is an integer congruent to $0, 1$, or 3 mod 4. For a specific example, take $a_1 = 1, a_2 = 2, a_3 = 3$ and $u = 3$. We can then take $b_{12} = 4, b_{13} = 7, b_{23} = 7, b'_{12} = 5, b'_{13} = 5, b'_{23} = 8$. In this example, the hypothesis of Theorem 3.10 concerning the g.c.d. of the $b_{i,j}$'s is also satisfied, so Theorem 3.10 gives us a pair of isospectral Riemannian nilmanifolds.

4. Curvature of the examples

We compare the curvature of the various examples of isospectral manifolds constructed in Sections 1-3. We continue to use the notation established in 1.2 and 3.1. Since the manifolds $G(j)$ are homogeneous, the curvature does not vary from point to point and thus can be viewed as a tensor on the vector space $\mathfrak{v} + \mathfrak{z}$ (i.e., on the Lie algebra $\mathfrak{g}(j)$, identified with the tangent space to $G(j)$ at the identity). The curvatures of the manifolds $M(j)$ in 1.2 and of the closed nilmanifolds $N(j, \Gamma)$ in 3.1 are the same as that of $G(j)$.

The curvature of $G(j)$ is easily computed. See [8] for details.

Proposition 4.1. *Given inner product spaces $\mathfrak{v}$ and $\mathfrak{z}$ and a linear map $j : \mathfrak{z} \to so(\mathfrak{v})$, let $G(j)$ be the associated Riemannian manifold constructed as in 1.2. Let $\{Z_1, \dots, Z_r\}$ be an orthonormal basis of $\mathfrak{z}$ and let $S = \frac{1}{2}\sum_{k=1}^{r} j^2(Z_k)$. For $X, Y \in \mathfrak{v}$ and Z, W in $\mathfrak{z}$ orthogonal unit vectors, the sectional curvature K and Ricci curvature are given as follows:*

(i)

$$K(X,Y) = -\frac{3}{4}\|[X,Y]\|^2,$$

$$K(X,Z) = \frac{1}{4}\|j(Z)X\|^2,$$

$$K(Z,W) = 0.$$

(ii)

$$Ric(X,Y) = \langle S(X), Y\rangle,$$

$$Ric(X,Z) = 0,$$

$$Ric(Z,W) = -\frac{1}{4}trace(j(Z)j(W)).$$

In particular, if j is injective, then the Ricci tensor is positive-definite on $\mathfrak{z}$ and negative semi-definite on $\mathfrak{v}$.

Corollary 4.2. *Fix inner product spaces $\mathfrak{v}$ and $\mathfrak{z}$, and let $j, j' : \mathfrak{z} \to so(\mathfrak{v})$ be injective linear maps. Let Ric and Ric′ denote the Ricci tensors of the associated manifolds $G(j)$ and $G(j')$. If $j \sim j'$, then*

$$Ric_{|\mathfrak{z}\times\mathfrak{z}} = Ric'_{|\mathfrak{z}\times\mathfrak{z}}.$$

Thus to compare the Ricci curvatures of the examples we need only look at $\mathrm{Ric}_{|\mathfrak{v}\times\mathfrak{v}}$. The eigenvalues of $\mathrm{Ric}_{|\mathfrak{v}\times\mathfrak{v}}$ are the eigenvalues of the operator S in Propositon 4.1.

Example 4.3. We first consider the 7-dimensional manifolds constructed in Example 1.10 (see also Example 3.5). We assume that T and T' are diagonal with respect to the standard basis of $\mathfrak{z}$ with diagonal entries (a, b, c) and $(-a, b, c)$, respectively. Then both $\mathrm{Ric}_{|\mathfrak{v}\times\mathfrak{v}}$ and $\mathrm{Ric}'_{|\mathfrak{v}\times\mathfrak{v}}$ are diagonalized by the standard basis of $\mathfrak{v}$. The four eigenvalues of $\mathrm{Ric}_{|\mathfrak{v}\times\mathfrak{v}}$ are all the expressions of the form $-\frac{1}{2}\{(1\pm a)^2+(1\pm b)^2+(1\pm c)^2\}$ with an even number of choices of minus signs in the terms in parentheses. The eigenvalues of Ric' are obtained by changing the sign of a; equivalently, they are all the expressions of the form above having an odd number of choices of minus signs.

Thus Examples 1.10 and 3.5 yield isospectral manifolds with different Ricci curvatures. We note, however, that the Ricci tensors have the same norm.

Example 4.4. We consider the continuous families of isospectral manifolds $G(j_u)$ constructed in Example 2.3, with $j_u = j_{a,b(u)}$. Let S_u be the operator associated with j_u as in Proposition 4.1. We have $S_u = \frac{1}{2}(a^2 + b(u)^2)$. As noted above, the manifolds $G(j_u)$, $u \in I$, have the same Ricci curvature if and only if the linear operators $S_u, u \in I$, are isospectral. An explicit computation shows that, for example, when a and b are chosen as in Example 3.11, then $\det(S_u)$ depends non-trivially on u. Thus the eigenvalues of the Ricci tensor of $G(j_u)$ (and of $M(j_u)$) depend non-trivially on u. In particular, the closed nilmanifolds in Example 3.11 have different Ricci curvature.

However, for all choices of a and b, $\mathrm{trace}({}^tS_uS_u)$ is independent of u. Consequently, the norm of the Ricci tensor does not change during any of the deformations.

Appendix

The proof of Proposition 3.7 is by an explicit calculation of the spectra. Using the Kirillov theory of representations of a nilpotent Lie group, Pesce [20] computed the eigenvalues of an arbitrary compact two-step nilmanifold. We first summarize his results.

Let $N = (\Gamma\backslash G, g)$ be a compact two-step nilmanifold. Thus G is a simply-connected two-step nilpotent Lie group, and g is a left-invariant metric on G. (We are temporarily dispensing with the notation established in the earlier sections.) Recall that the Laplacian of N is given by $\Delta = -\sum_i X_i^2$, where $\{X_1, X_2, \ldots, X_n\}$ is an orthonormal basis of the Lie algebra $\mathfrak{g}$ relative to the inner product $<\,,\,>$ defined by g. Let $\rho = \rho_\Gamma$ denote the right action of G on $L^2(N)$; then the Laplacian acts on $L^2(N)$ as $\Delta = -\sum_i \rho_* X_i^2$.

Given any unitary representation (V, π) of G (here V is a Hilbert space and π is a representation of G on V), we may define a Laplace operator $\Delta_{g,\pi}$ on V by $\Delta_{g,\pi} = -\sum_i \pi_* X_i^2$. The eigenvalues of this operator depend only on g and the equivalence class of the representation π. The space $(L^2(N), \rho)$ is the countable direct sum of irreducible representations (V_α, π_α), each occurring with finite multiplicity. The spectrum of N is the union, with multiplicities, of the spectra of the operators Δ_{g,π_α}.

Kirillov [18] showed that the equivalence classes of irreducible unitary representations of the simply-connected nilpotent Lie group G are in one to one correspondence with the orbits of the co-adjoint action of G on the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G. We will denote the representation corresponding to the co-adjoint orbit of $\sigma \in \mathfrak{g}^*$ by π_σ.

Richardson [23] computed the decomposition of $L^2(\Gamma\backslash G)$ into irreducible representations π_σ for an arbitrary compact nilmanifold. In case G is two-step nilpotent, this decomposition can be given very explicitly.

Notation A.1. Given $\sigma \in \mathfrak{g}^*$, define $B_\sigma : \mathfrak{g} \times \mathfrak{g} \to R$ by

$$B_\sigma(X, Y) = \sigma([X, Y]).$$

Let $\mathfrak{g}^\sigma = \ker(B_\sigma)$ and let $\overline{B_\sigma}$ be the non-degenerate skew-symmetric bilinear form induced by B_σ on $\mathfrak{g}/\mathfrak{g}^\sigma$. The image of $\log(\Gamma)$ in $\mathfrak{g}/\mathfrak{g}^\sigma$ is a lattice, which we denote by $\mathcal{A}_\sigma$.

We will write $\Delta_{g,\sigma}$ for Δ_{g,π_σ}.

Proposition A.2. (See [20].) *Let $N = (\Gamma\backslash G, g)$ be a compact two-step nilmanifold, let $\mathfrak{g}$ be the Lie algebra of G, and let $\sigma \in \mathfrak{g}^*$. Then π_σ appears in the quasi-regular representation ρ_Γ of G on $L^2(N)$ if and only if $\sigma(\log(\Gamma) \cap \mathfrak{g}^\sigma) \subset \mathbf{Z}$. In this case the multiplicity of π_σ is $m_\sigma = 1$*

if $\sigma(\mathfrak{z}) = \{0\}$*, and* $m_\sigma = (det\overline{B_\sigma})^{1/2}$ *otherwise, where the determinant is computed with respect to a lattice basis of* $\mathcal{A}_\sigma$*.*

Let $<,>$ be the inner product on $\mathfrak{g}^*$ defined by the Riemannian inner product on $\mathfrak{g}$.

Proposition A.3. [20] *Let* $\mathfrak{z} = [\mathfrak{g}, \mathfrak{g}]$*.*

(a) If $\sigma(\mathfrak{z}) = 0$*, then* π_σ *is a character of* G *and*

$$spec(\Delta_{g,\sigma}) = \{4\pi^2\|\sigma\|^2\}.$$

(b) If $\sigma(\mathfrak{z}) \neq \{0\}$*, let* $\pm(-1)^{1/2}d_1, \dots, \pm(-1)^{1/2}d_r$ *be the eigenvalues of* $\overline{B_\sigma}$*. Then*

$$spec(\Delta_{g,\sigma}) = \{\mu(\sigma, p, g) : p \in \mathbf{N}^r\},$$

where

$$\mu(\sigma, p, g) = 4\pi^2 \sum_{i=1,\dots l} \sigma(Z_i)^2 + 2\pi \sum_{k=1,\dots,r} (2p_k + 1)d_k$$

with $\{Z_1, \dots, Z_l\}$ *a* g*-orthonormal basis of* $\mathfrak{g}^\sigma$*. The multiplicity of an eigenvalue* μ *is the number of* $p \in \mathbf{N}^r$ *such that* $\mu = \mu(\sigma, p, g)$*.*

Proof of Proposition 3.7. We use the notation of 3.1, 3.6 and A.1, and let $G = G(j)$ and $N = N(j, \Gamma)$. By an elementary and standard argument, the part of spec(N) corresponding to all the characters π_σ in part (a) of Proposition A.3 coincides with spec$(N_0(j, \Gamma))$. Thus we need only consider those representations π_σ with $\sigma(\mathfrak{z}) \neq 0$.

For z as in Proposition 3.7 and $x, y \in \mathfrak{g} = \mathfrak{g}(j)$, observe that

$$B_\sigma(x, y) =< [x, y], z > \sigma(z) =< j(z)x, y > \sigma(z) \tag{1}$$

by 1.1. Thus $\mathfrak{g}^\sigma$, as defined in A.1, coincides with $\mathfrak{a} + \mathfrak{z}$. Hence the occurrence condition for π_σ in Proposition A.2 just says that $\sigma_{|\mathfrak{a}+\mathfrak{z}} \in \mathcal{K}^*$. Observe that $\mu \in \mathfrak{g}^*$ lies in the same co-adjoint orbit as σ if and only if $\mu_{|\mathfrak{a}+\mathfrak{z}} = \sigma_{|\mathfrak{a}+\mathfrak{z}}$; therefore we may identify co-adjoint orbits with elements σ of $\mathcal{K}^*$.

By equation (1), the eigenvalues of B_σ, and thus of $\overline{B_\sigma}$, are determined by the eigenvalues of $j(z)$ and by $\sigma(z)$. Moreover, for $\{Z_1 \dots Z_l\}$ an orthonormal basis of $\mathfrak{g}^\sigma = \mathfrak{a}+\mathfrak{z}$, we have $\sum_{i=1}^l \sigma(Z_i)^2 = \|\sigma\|^2$. Hence by Proposition A.3(b), the eigenvalues of $\Delta_{g,\sigma}$ are completely determined by the data in (ii) and (iii) of Proposition 3.7.

It remains to show that the data (i)–(iii) determine the multiplicity m_σ of π_σ in the representation ρ_Γ of G on $L^2(N)$. First observe that the center $z(G)$ has Lie algebra $\mathfrak{g}^\sigma = \mathfrak{a} + \mathfrak{z}$. Let $\pi : G \to G/z(G)$ be the projection. The group $\pi(G)$ with the Riemannian structure induced by that of G is Euclidean and $T := \pi(G)/\pi(\Gamma)$ is a flat torus. Let $\overline{\mathfrak{g}} = \mathfrak{g}/\mathfrak{g}^\sigma$ and letting $\mathcal{A}_\sigma$ be as in A.1; then the Lie group exponential map from $\overline{\mathfrak{g}}$ to $\pi(G)$ carries $\mathcal{A}_\sigma$ to $\pi(\Gamma)$ and induces an isometry from the torus $\overline{\mathfrak{g}}/\mathcal{A}_\sigma$ to T, where $\overline{\mathfrak{g}}$ is given the inner product induced by that on $\mathfrak{g}$. (Note that $\overline{\mathfrak{g}}$ may be identified with the subspace $\mathfrak{u}$ of $\mathfrak{g}$ defined in 3.6.) N fibers over T as a Riemannian submersion with fiber $z(G)/(z(G) \cap \Gamma)$. The fiber is isometric to the torus $(\mathfrak{a} + \mathfrak{z})/\mathcal{K}$.

Now consider the multiplicity m_σ of π_σ, given in Proposition A.2. By equation (1), the determinant of $\overline{B_\sigma}$ with respect to an orthonormal basis of $\mathfrak{g}/\mathfrak{g}^\sigma$ (relative to the induced inner product defined above) is determined by the eigenvalues of $j(z)$ and by $\sigma(z)$. To find the determinant with respect to a lattice basis of $\mathcal{A}_\sigma$, the only additional information needed is the volume of T (i.e., the "volume" of the lattice). Thus it remains to show that the volume of T is determined by the data (i)–(iii).

We have two ways of viewing N as a principal torus bundle over a torus; both are Riemannian submersions. First we have the submersion discussed alone:

$$\begin{array}{ccc} z(G)/(z(G) \cap \Gamma) & \longrightarrow & N \\ & & | \\ & & T \end{array}$$

Secondly we have a submersion with circle fiber:

$$\begin{array}{ccc} S := [G,G]/([G,G] \cap \Gamma) & \hookrightarrow & N \\ & & | \\ & & N_0 \end{array}$$

The fiber circle is isometric to $\mathfrak{z}/\mathcal{K} \cap \mathfrak{z})$ with the inner product $<\,,\,>$.

The second fibration and the data (i)–(iii) enable us to determine $\mathrm{vol}(N)$. Indeed (i) gives us $\mathrm{vol}(N_0)$, and from (iii) we can determine the length of the circle S as follows: From (iii) we can find $\min\{\|\sigma\| : \sigma \in \mathcal{K}^*$ and $\|\sigma\| = |\sigma(z)|\} = \min\{\|\sigma\| : \sigma \in \mathcal{K}^*$ and $\sigma_{|\mathfrak{a}} = 0\}$; call this c. But c is precisely the length of a basis element of the lattice in $\mathfrak{z}^*$ dual to $\mathcal{K} \cap \mathfrak{z} = \log(\Gamma) \cap \mathfrak{z}$. Hence c determines the length of the circle S. We conclude that the data (i)–(iii) determine $\mathrm{vol}(N)$.

Next the second half of the data in (iii), i.e., $\{\|\tau\| : \tau \in \mathcal{K}^*\}$ determines the spectrum of the fiber torus in the first submersion and thus the volume of the fiber. This together with vol(N) determines vol(T). Thus the multiplicity m_σ is determined by the data (i)–(iii). This completes the proof.

Note added in proof: R.Gornet, D. Schueth, D. Webb and the authors recently showed that the boundaries of the manifolds in Theorem 0.1 are isospectral. This construction yields continuous families of isospectral closed manifolds which have no common covering and which are not locally homogeneous. Z. Szabó independently and simultaneously constructed pairs of isospectral closed manifolds with these properties; these are described in a revised version of his article [25].

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